- $\hat{a}$ annihilation (or lowering) operator
- $\hat{a}^+$ Creation (or raising) operator

Since there is no energy below the ground state, we should have:

$$\hat{a}\psi_n \longrightarrow E_n$$
$$\hat{a}\psi_n = \psi_{n-1} \longrightarrow E_{n-1} = E_n - \hbar\omega$$
$$\hat{a}\psi_{n-1} = \psi_{n-2} \longrightarrow E_{n-2} = E_n - 2\hbar\omega$$
$$\vdots$$
$$\hat{a}\psi_1 = \psi_0 \longrightarrow E_0 = E_1 - \hbar\omega$$
$$\hat{a}\psi_0 = 0 \longrightarrow \text{No energy}$$

we can use this to find all $\psi_n$'s:

$$\hat{a}\psi_0 = 0 \rightarrow \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right)\psi_0 = 0$$

$$\left(x + \frac{i}{m\omega}\left(-i\hbar\frac{\partial}{\partial x}\right)\right)\psi_0 = 0$$

$$\frac{\hbar}{m\omega}\psi_0' + x\psi_0 = 0$$

$$\psi_0' + \frac{m\omega}{\hbar}x\psi_0 = 0$$

Let's try solution of the form $e^{ax^n} \rightarrow$

$$anx^{n-1}e^{ax^n} + \frac{m\omega}{\hbar}xe^{ax^n} = 0$$

$$anx^{n-1} + \frac{m\omega}{\hbar}x = 0 \quad \text{must be for all } x \Rightarrow n = 2$$

$$2a + \frac{m\omega}{\hbar} = 0 \Rightarrow a = -\frac{m\omega}{2\hbar} \Rightarrow$$

$$\psi_0 = Ae^{-\frac{m\omega}{2\hbar}x^2}$$

Find $A$ by normalizing $A$:

$$\int \psi_0^* \psi_0 \, dx = 1 \rightarrow A^2 \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2} dx = 1$$

$$u = \left(\frac{m\omega}{\hbar}\right)^{1/2} x \rightarrow du = \left(\frac{m\omega}{\hbar}\right)^{1/2} dx \rightarrow$$

$$A^2 \left(\frac{\hbar}{m\omega}\right)^{1/2} \underbrace{\int_{-\infty}^{\infty} e^{-u^2} du}_{\sqrt{\pi}} = 1$$

$$A^2 \left(\frac{\hbar}{m\omega}\right)^{1/2} \sqrt{\pi} = 1$$

where we have used the Gaussian integral:

$$\Rightarrow A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\boxed{\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}$$

Even parity

(Symmetry $\Rightarrow$ even or odd parity)

Let's find the Ground State energy now:

$$\hat{H}\psi_0 = E_0 \psi_0$$

$$-\frac{\hbar^2}{2m} \underbrace{\frac{d^2}{dx^2}}_{?} \psi_0 + \frac{m\omega^2}{2} \psi_0 = E_0 \psi_0$$

$$\psi_0 = A e^{-\frac{m\omega}{2\hbar}x^2} \rightarrow \psi_0' = A \frac{-m\omega}{2\hbar}(2x) e^{-\frac{m\omega}{2\hbar}x^2} \rightarrow$$

$$\psi_0'' = \frac{-Am\omega}{\hbar} e^{-\frac{m\omega}{2\hbar}x^2} + A\left(\frac{m\omega}{\hbar}\right)^2 x^2 e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \left[-\frac{m\omega}{\hbar} + \left(\frac{m\omega}{\hbar}\right)^2 x^2\right] \psi_0$$

$$\Rightarrow \quad -\frac{\hbar^2}{2m}\left(-\frac{m\omega}{\hbar} + \frac{m^2\omega^2}{\hbar^2}x^2\right)\cancel{\psi_0} + \frac{m\omega^2}{2}\cancel{\psi_0} = E_0\cancel{\psi_0}$$

$$\frac{\hbar\omega}{2} - \frac{m\omega^2}{2} + \frac{m\omega^2}{2} = E_0 \quad \Rightarrow$$

$$\boxed{E_0 = \frac{\hbar\omega}{2}}$$

also had ;

$$\boxed{\psi_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}$$